Physics 281

[redacted] Wakes

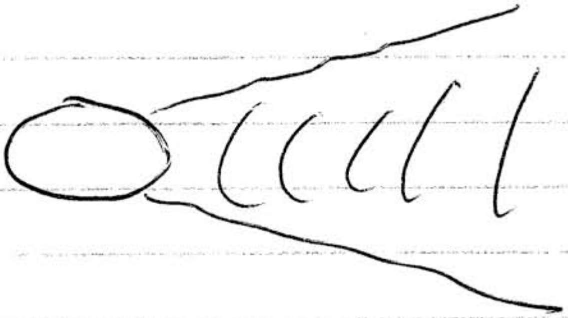
→ What ?

c.f. { Prandtl-Tietjens, Falkovich, L-L

Wake is :

- region of departure from potential flow, behind object moving thru water and experiencing drag.

i.e.



wake.

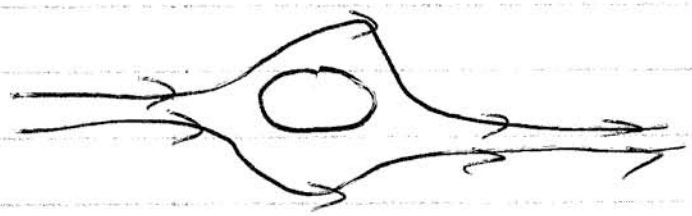
- wake is inexorably coupled to drag

i.e. → drag ≡ thinking in frame where object at rest, drag results from loss of flow momentum to object.

→ wake is region of flow where loss of momentum is evident.

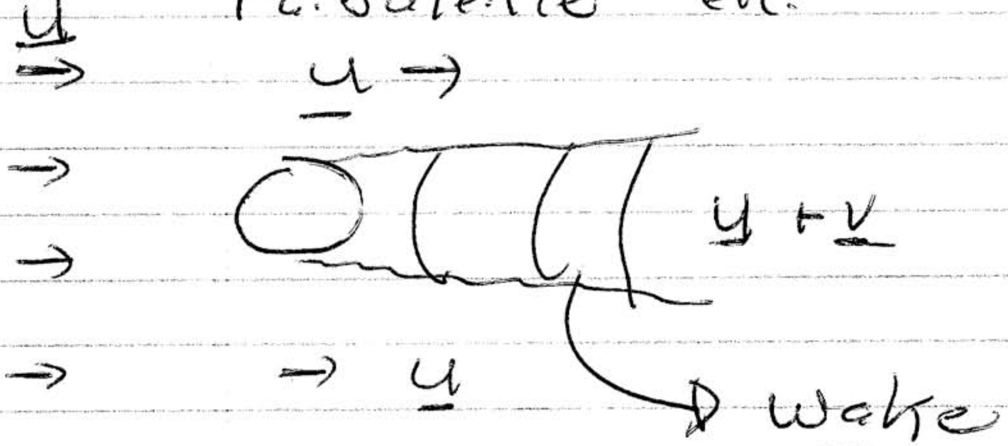
i.e.,

- if potential flow (no drag)



symmetry upstream downstream in ⊥ displacement of fluid element

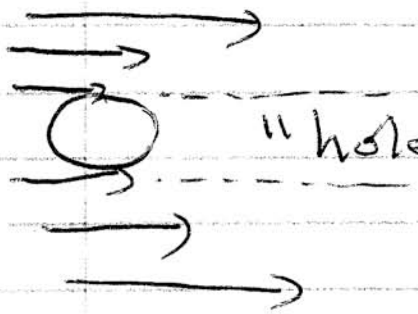
- with no-slip b.c., viscosity, turbulence etc.



$\underline{V}$ opposite $\underline{u}$ i.e.

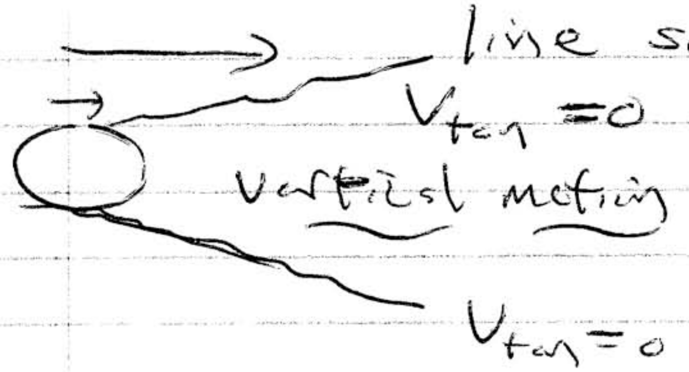
$$\begin{cases} u > 0 \\ v < 0 \end{cases}$$

– origin of wake is no-slip b.c. + $\left\{ \begin{array}{l} \text{viscosity,} \\ \text{turbulence} \end{array} \right.$, after separation



but flow is unstable!

oo



$\underline{\omega} = \underline{\nabla} \times \underline{V} \neq 0$

– boundary of wake traced by fluid particles:

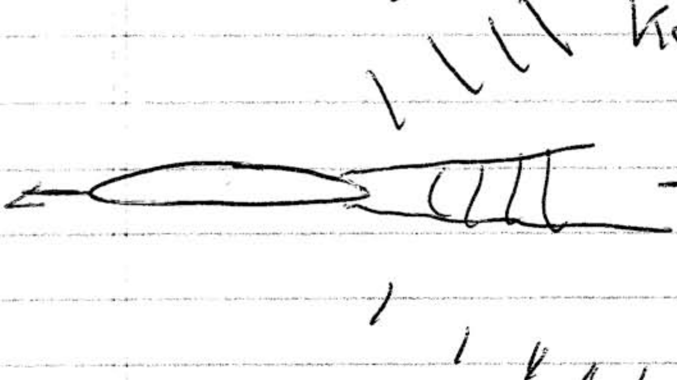
→ passing close to body,

→ scattered by diffusion and turbulent mixing

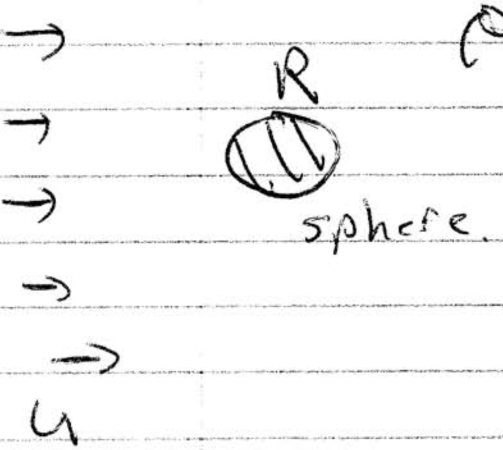
⇒ expansion

Note:

- in general, wake multi-component



- here, consider spherical cow of wake problems



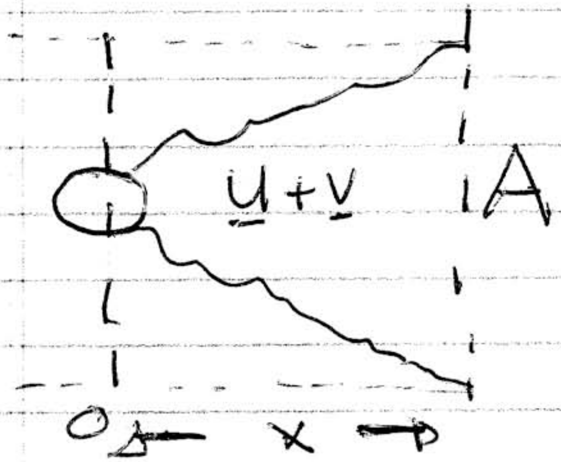
so $F_d \sim \rho U^2 R^2 f(Re)$

→ no surface effects.

→ How calculate wake structure?

Force of Drag ≡ Net Rate of Momentum Loss from Flow!

i.e.



downstream distance

Rate Momentum Loss $\equiv$

$$A\, P_{total}(x) - A\, P_{Tot}(0) = F_d$$

$$P_{Tot}(0) \cong P + \rho U^2/2$$

total head.

$$P_{tot}(x) \cong P + \rho(U+V)^2/2$$

$$A \sim \pi\, w(x)^2$$

$w \equiv$ width of wake at x downstream

so

$$F_d \approx w(x)^2 \left[\left(P + \frac{\rho}{2}(U+V)^2 \right) - \left(P + \frac{\rho U^2}{2} \right) \right]$$

So

$$F_d \sim w(x)^2 \left[\cancel{p} + \cancel{\frac{\rho U^2}{2}} + \frac{2\rho U V_x}{2} - \cancel{p} - \cancel{\frac{\rho U^2}{2}} \right]$$

$$\sim \rho U V_x \, w(x)^2$$

n.b. why $p(0) \sim p(x)$?

$$\therefore \boxed{F_d \sim \rho U V_x \, w(x)^2}$$

Now, need $w(x)$ to get V_x !

→ Observe:

- problem now reduced to one of scale
- wakes are self-similar!

$\Rightarrow w \sim x^{\alpha}$, α ?

- wakes can be laminar [redacted] or turbulent

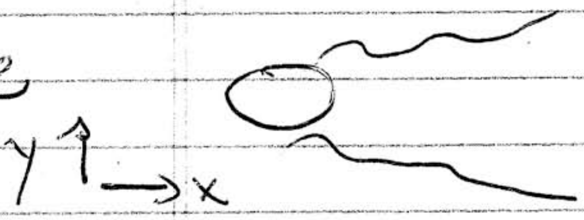
(i) Laminar $\qquad UR/\nu < 1$

now $\quad \cancel{\frac{\partial \underline{v}}{\partial t}} + \underline{v}\cdot\nabla\underline{v} - \nu\nabla^2\underline{v} = -\frac{\nabla p}{\rho}$

st. state

$$\underline{U}\cdot\nabla\underline{v} + \cancel{\underline{v}\cdot\nabla\underline{v}} - \nu\nabla^2\underline{v} = -\frac{\nabla p}{\rho}$$

so



$\nabla\cdot\underline{v} = 0$

$$U\partial_x v_y - \nu(\cancel{\partial_x^2}+\partial_y^2)v_y = -\frac{\partial_y p}{\rho}$$

and

$$U\partial_x v_x - \nu\partial_y^2 v_x = -\frac{\partial_x p}{\rho}$$

take $\quad \partial_x \sim 1/X$

$\qquad\quad \partial_y \sim 1/W$

$$\left(\frac{U}{x} - \frac{\nu}{w^2}\right) v_y \sim -\frac{p}{w\rho}$$

$$\left(\frac{U}{x} - \frac{\nu}{w^2}\right) v_x \sim -\frac{p}{x\rho}$$

$$\underline{\nabla} \cdot \underline{v} = 0 \quad \Rightarrow \quad \frac{v_x}{x} \sim \frac{v_y}{w}$$

as p negligible (will show) $\Rightarrow$

$$\frac{U}{x} \sim \frac{\nu}{w^2}$$

$$\Rightarrow \boxed{w \sim (\nu x / U)^{1/2}}$$

$\rightarrow$ diffusive spreading of momentum by ν

$\rightarrow \sim (\nu t)^{1/2}$ with $t \sim x/U$.

so
$$W \sim \left(\frac{x}{R}\right)^{1/2}\left(\frac{\nu R}{U}\right)^{1/2}$$

$$\boxed{W/R \sim \left(\frac{x}{R}\right)^{1/2} 1/Re^{1/2}}$$

$$\boxed{V_x \sim \frac{F_d}{\rho U W^2}}$$

→ akin Blasius B.L. thickness.

→ in case you are wondering:

$\hat{y}$: $\frac{P}{\rho W} \sim \frac{\nu V_y}{W^2}$ (if assume)

and
$$\frac{V_x}{x} \sim \frac{V_y}{W}$$

$$\Rightarrow P \sim \rho \nu V_x / x$$

and $P/\rho x \sim \frac{\nu V_x}{x^2} \ll \frac{\nu V_x}{W^2}$

drop P. and safely $\nu V_y / W^2$

(ii) Turbulent $\qquad Re \sim UR/\nu \gg 1$

$$\underline{U}\cdot\nabla\underline{V} + \underline{V}\cdot\nabla\underline{V} - \cancel{\nu\nabla^2\underline{V}} = -\frac{\nabla p}{\rho}$$

ignore

$$\Rightarrow \quad \underbrace{\frac{U}{X}V_x \sim \frac{\tilde{V}_y}{W}V_x}_{\text{wave spreads by advection, not diffusion}}$$

$\tilde{V}_y \sim$ turbulent velocity $\qquad \boxed{W \sim \frac{\tilde{V}_y}{U}x}$

Take wake turbulence isotropic;

so $\tilde{V}_x \sim \tilde{V}_y$. $\qquad$ Fair? Test?

$$W \sim x\tilde{V}_x/U$$

but from drag:

$$\tilde{V}_x \sim F_d/\rho U W^2$$

$\Rightarrow$

$\underline{\underline{\mathcal{S}}}$

$$W \sim x \frac{F_d}{\rho u^2 w^2} \sim x \left(F_d / \rho u^2 w^2 \right)$$

$$W^3 \sim F_d\, x / \rho u^2$$

$$\Rightarrow \boxed{\begin{aligned} W &\sim \left(F_d / \rho u^2 \right)^{1/3} x^{1/3} \\ &\sim \left(C_D R^2 \right)^{1/3} x^{1/3} \end{aligned}}$$

<u>then, comparing widths:</u>

$$\boxed{\begin{aligned} &\text{laminar:} \quad w/R \sim (x/R)^{1/2}\, Re^{-1/2} \\ &\qquad\qquad Re \sim UR/\nu \\ &\text{turbulent:} \quad w/R \sim (x/R)^{1/3}\, C_D^{1/3} \end{aligned}}$$

interestingly <u>Laminar</u> wake expands with downstream length more rapidly!

Why?

→ turbulence can relax ∇V behind object (due separation) rapidly, and faster than ν. Thus surrounding flow penetrates the dead water region more rapidly.

Also observe: Wake Re drops with x.

$\Rightarrow$

$$Re \sim \frac{W V_y}{\nu} \sim \frac{W V_x}{\nu} \sim \frac{\cancel{W}}{\nu} \frac{F_d}{\rho U W^{\cancel{2}}}$$

↑ y direction (spr) ↑ wake flow Re

$$Re \sim F_d / \rho U W \nu$$

$$\sim \frac{U^{\cancel{2}} R^2 \cancel{\rho_0} C_D}{\nu \cancel{\rho_0} \cancel{U} (C_D R^2)^{1/3} x^{1/3}}$$

$C_D \sim 1$

$$\sim \left(\frac{UR}{\nu}\right) \left(R/x\right)^{1/3}$$

$$\boxed{Re(x) \sim Re_0 \, (R/x)^{1/3}}$$

and $Re(x) \to 1$ at

$$x_L \sim R\,(Re_0)^3$$

distance behind boat where turbulent wake transitions to laminar.

i.e. akin l_d: transition from turbulent mixing to viscous mixing

N.B. In wake, vortical/rotational region can expand into irrotational region, but never reverse!

i.e. would really violate H-Thm ...